

Models of relativistic particle with curvature and torsion revisited

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Abstract

Models, describing relativistic particles, where Lagrangian densities depend linearly on both the curvature and the torsion of the trajectories, are revisited in $D = 3$ space forms. The moduli spaces of trajectories are completely and explicitly determined using the Lancret program. The moduli subspaces of closed solitons in the three sphere are also determined.

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1 Some background

The conventional approach to consider Lagrangians that describe relativistic particles is based on certain extensions of the original space-time by extra variables that provide the required new degrees of freedom. Recently however, a new approach appeared in the literature (see for example [1, 6, 9, 10, 11, 13, 14, 15] and references therein). In this setting, the particle systems are described by Lagrangians that, being formulated in the original space-time (so they are intrinsic), in return for they depend on higher derivatives. Therefore, the attractive point in this new philosophy is that the spinning degrees of freedom are assumed to be encoded in the geometry of the trajectories. Finally, the Poincaré and invariance requirements imply that the admissible Lagrangian densities must depend on the extrinsic curvatures of the curves in the background gravitational field.

Most of the published papers, in this direction, involve actions that only depend on the first curvature of trajectories (the curvature, which plays the role of proper acceleration of the particle). However, it seems important to investigate models of particles with curvature and torsion.

Along this note, $M(C)$, will denote a three dimensional space with constant curvature C . In a suitable space of curves, Λ in $M(C)$, (for example the space of closed curves or that of curves satisfying certain second order boundary data, clamped curves), we have a three-parameter family of actions, $\{\mathcal{F}_{mnp} : \Lambda \rightarrow \mathbb{R} : m, n, p \in \mathbb{R}\}$, defined by

$$\mathcal{F}_{mnp}(\gamma) = \int_{\gamma} (m + n\kappa + p\tau) ds, \quad (1)$$

where s , κ and τ stand for the arclength parameter, curvature and torsion of γ , respectively.

The main purpose of this note is to determine, explicitly and completely, the moduli space of trajectories in the particle model $[M(C), \mathcal{F}_{mnp}]$. In particular, we provide algorithms to obtain the trajectories of a given model. The closed trajectories, when there exist, are also obtained from an interesting quantization principle.

It should be noticed that, this problem was considered in [6] when $C = 0$, flat space. In that paper, the authors showed that trajectories are helices (that is curves with both curvature and torsion being constant) in $M(0)$. However, this is not true. In fact, we prove here that trajectories in the model $[M(0), \mathcal{F}_{0np}]$ are curves of Lancret with slope determined from the values of n and p . To understand this note better, we recall in the next section the nice geometry of the Lancret curves not only in classical setting, $C = 0$, but also when C is arbitrary.

2 The extended Lancret program

A curve of Lancret (or general helix) in \mathbb{R}^3 is a curve with constant slope, that is, one whose tangent makes a constant angle with a fixed straight line (the axis of the general helix). In other words, the tangent indicatrix of a curve of Lancret lies in a plane of \mathbb{R}^3 . The two main statements in the theory of these curves are

1. A classical result stated by M.A.Lancret in 1802 and first proved by B. de Saint Venant in 1845 (see [17] for details) which gives an algebraic characterization for Lancret's curves. *The curves of Lancret are those curves that the ratio of curvature and torsion is constant.*

2. The geometric approach to the problem of *solving natural equations* for general helices in \mathbb{R}^3 . *A curve in \mathbb{R}^3 is a Lancret one if and only if it is a geodesic of a right cylinder on a plane curve.*

Notice that this class of curves includes not only those curves with torsion vanishing identically but also the ordinary helices (helices) which have both torsion and curvature being nonzero constants. These cases correspond, in the geometric approach, with geodesics of right cylinders shaped on plane curves with constant curvature. A plane with directrix being a straight line and a circular right cylinder determined by a circle, respectively. We will refer these two cases as trivial Lancret curves.

In [4], the second author used the concept of *Killing vector field along a curve* to define the notion of general helix in a three dimensional real space form, $M(C)$. Then, he obtained the extension of the Lancret program to this framework. It is a bit more subtle than one might suppose a priori, as evidenced by the difference between the spherical and the hyperbolic cases. In fact, while the former case is nicely analogous to the Euclidean one, the later only presents trivial Lancret curves. To be more precise

1. A curve in the hyperbolic space, \mathbb{H}^3 , is a general helix if and only if either its torsion vanishes identically or it is an ordinary helix. That is, the class of Lancret curves in a hyperbolic space is just reduced to that of ordinary Lancret curves.
2. A curve in the sphere, \mathbb{S}^3 with constant curvature C , is a general helix if and only if either its torsion vanishes identically or the curvature, κ , and the torsion, τ , are related by $\tau = a\kappa \pm \sqrt{C}$, where a is a certain constant which will be interpreted as a kind of slope.

In that paper, the solving natural equations is obtained as follows. *A curve in \mathbb{S}^3 is a general helix if and only if it is a geodesic of a Hopf cylinder.* That is a surface obtained when one makes the complete lifting, via the usual Hopf map, of a curve in the corresponding round two sphere.

However, the *closed curve problem* for general helices in \mathbb{S}^3 was also given by taking advantage from the well known isometry type of the Hopf tori obtained on closed directrices in the two sphere and it gives a very deep difference with respect to the classical setting.

The main result of this note can be stated as follows

A curve $\gamma \in \Lambda$ is a critical point of \mathcal{F}_{mnp} if and only if γ is a Lancret curve in $M(C)$. In other words, the spinning relativistic particles in the model $[M(C), \mathcal{F}_{mnp}]$ evolve along Lancret curves of $M(C)$.

3 The field equations

The metric of $M(C)$ will be denoted by $g = \langle, \rangle$ and its Levi-Civita connection by ∇ . Let $\gamma = \gamma(t) : I \subset \mathbb{R} \rightarrow M(C)$ be an immersed curve with speed $v(t) = |\gamma'(t)|$, curvature κ , torsion τ and Frenet frame $\{T, N, B\}$. Then, one can write the Frenet equations of γ as

$$\begin{aligned}\nabla_T T &= \kappa N, \\ \nabla_T N &= -\kappa T - \tau B, \\ \nabla_T B &= \tau N.\end{aligned}$$

In order to derive first variation formulas for \mathcal{F}_{mnp} , we will use the following standard terminology (see [7] for details). For a curve $\gamma : [0, L] \rightarrow M$, we take a variation, $\Gamma = \Gamma(t, r) : [0, L] \times (-\varepsilon, \varepsilon) \rightarrow M$ with $\Gamma(t, 0) = \gamma(t)$. Associated with this variation is the variation vector field $W = W(t) = \frac{\partial \Gamma}{\partial r}(t, 0)$ along the curve $\gamma(t)$. We also write $V = V(t, r) = \frac{\partial \Gamma}{\partial t}(t, r)$, $W = W(t, r)$, $v = v(t, r)$, $T = T(t, r)$, $N = N(t, r)$, $B = B(t, r)$, etc., with the obvious meanings. We let s denote arclength, and put $V(s, r)$, $W(s, r)$ etc., for the corresponding reparametrizations. To obtain the formulas without doing tedious computations, we quote general formulas for the variations of v , κ and τ in γ and in the direction of W . These are obtained using standard computations that involve the Frenet equations

$$\begin{aligned}W(v) &= \langle \nabla_T W, T \rangle v, \\ W(\kappa) &= \langle \nabla_T^2 W, N \rangle - 2\kappa \langle \nabla_T W, T \rangle + C \langle W, N \rangle, \\ W(\tau) &= \left(\frac{1}{\kappa} \langle \nabla_T^2 W + CW, B \rangle \right)_s + \tau \langle \nabla_T W, T \rangle + \kappa \langle \nabla_T W, B \rangle,\end{aligned}$$

where the subscript s denotes differentiation with respect to the arclength.

Now, we use a standard argument which involves the above obtained formulas and some integrations by parts to get the variation of \mathcal{F}_{mnp} along γ in the direction of W

$$\delta \mathcal{F}_{mnp}(\gamma)[W] = \int_{\gamma} \langle \Omega(\gamma), W \rangle ds + [\mathcal{B}(\gamma, W)]_0^L, \quad (2)$$

where $\Omega(\gamma)$ and $\mathcal{B}(\gamma, W)$ stand for the Euler-Lagrange and Boundary operators, respectively, and they are given by

$$\Omega(\gamma) = (-m\kappa + p\kappa\tau - n\tau^2 + nC)N + (p\kappa_s - n\tau_s)B,$$

$$\begin{aligned}\mathcal{B}(\gamma, W) &= -\frac{p}{\kappa} \langle \nabla_T^2 W, B \rangle + \langle \nabla_T W, N \rangle \\ &+ m \langle W, T \rangle + \left(n\tau - \frac{pC}{\kappa} - p\kappa \right) \langle W, B \rangle.\end{aligned}$$

Proposition 1 (Second order boundary conditions) *Given $q_1, q_2 \in M$ and $\{x_1, y_1\}, \{x_2, y_2\}$ orthonormal vectors in $T_{q_1}M$ and $T_{q_2}M$ respectively, define the space of curves*

$$\Lambda = \{\gamma : [t_1, t_2] \rightarrow M : \gamma(t_i) = q_i, T(t_i) = x_i, N(t_i) = y_i, 1 \leq i \leq 2\}.$$

Then, the critical points of the variational problem $\mathcal{F}_{mnp} : \Lambda \rightarrow \mathbb{R}$ are characterized by the following Euler-Lagrange equations

$$-m\kappa + p\kappa\tau - n\tau^2 + nC = 0, \quad (3)$$

$$p\kappa_s - n\tau_s = 0. \quad (4)$$

Proof. Let $\gamma \in \Lambda$ and $W \in T_\gamma\Lambda$, then W defines a curve in Λ associated with a variation $\Gamma = \Gamma(t, r) : [0, L] \times (-\varepsilon, \varepsilon) \rightarrow M$ of $\gamma, \Gamma(t, 0) = \gamma(t)$. Therefore, we can make the following computations along Γ

$$\begin{aligned}W &= d\Gamma(\partial_r), \\ \nabla_T W &= fT + d\Gamma(\partial_r T), \\ \nabla_T^2 W &= (\partial_s f + f)T + (\kappa f + \partial_r \kappa)N + \kappa d\Gamma(\partial_r N) + R(T, W)T,\end{aligned}$$

here $f = \partial_r(\log v)$. Then, we evaluate these formulas along the curve γ by making $r = 0$ and use the second order boundary conditions to obtain the following values at the endpoints

$$\begin{aligned}W(t_i) &= 0, \\ \nabla_T W(t_i) &= f(t_i)x_i, \\ \nabla_T^2 W(t_i) &= (\partial_s(f) + f)(t_i)x_i + (\kappa f + \partial_r \kappa)(t_i)y_i.\end{aligned}$$

As a consequence,

$$[\mathcal{B}(\gamma, W)]_{t_1}^{t_2} = 0.$$

Then, γ is a critical point of the variational problem $\mathcal{F}_{mnp} : \Lambda \rightarrow \mathbb{R}$, that is $\delta\mathcal{F}_{mnp}(\gamma)[W] = 0$, for any $W \in T_\gamma\Lambda$ if and only if $\Omega(\gamma) = 0$ which gives (3) and (4).

4 The moduli spaces of trajectories

The field equations, (3,4), can be nicely integrated. The set of solutions is summarized by the following three tables which correspond with Euclidean, hyperbolic and spherical case, respectively. All the solutions are Lancet curves. Similarly to the Euclidean case, curves with zero torsion, including geodesics, and helices are considered as special cases of Lancet curves(trivial Lancet curves). For simplicity of interpretation, we have represented different cases according with the values of the three parameters that define the action.

m	n	p	Solutions in \mathbb{R}^3, $C = 0$
$\neq 0$	$= 0$	$= 0$	Geodesics $\kappa = 0$
$= 0$	$= 0$	$\neq 0$	Circles κ constant and $\tau = 0$
$= 0$	$\neq 0$	$= 0$	Plane curves $\tau = 0$
$\neq 0$	$\neq 0$	$= 0$	Helices with $\kappa = \frac{-n\tau^2}{m}$
$\neq 0$	$= 0$	$\neq 0$	Helices with arbitrary κ and $\tau = \frac{m}{p}$
$= 0$	$\neq 0$	$\neq 0$	Lancet curves with $\tau = \frac{p}{n}\kappa$
$\neq 0$	$\neq 0$	$\neq 0$	Helices with $\kappa = \frac{-na^2}{m+ap}$, $\tau = \frac{ma}{m+ap}$ and $a \in \mathbb{R} - \{-\frac{m}{p}\}$

m	n	p	Solutions in \mathbb{H}^3, $C = -c^2$
$\neq 0$	$= 0$	$= 0$	Geodesics $\kappa = 0$
$= 0$	$= 0$	$\neq 0$	Curves with κ constant and $\tau = 0$
$= 0$	$\neq 0$	$= 0$	Do not exist
$\neq 0$	$\neq 0$	$= 0$	Helices with $\kappa = \frac{-n(c^2+\tau^2)}{m}$
$\neq 0$	$= 0$	$\neq 0$	Helices with arbitrary κ and $\tau = \frac{m}{p}$
$= 0$	$\neq 0$	$\neq 0$	Helices with $\kappa = \frac{-n(c^2+a^2)}{ap}$ and $\tau = -\frac{c^2}{a}$ and $a \in \mathbb{R} - \{0\}$
$\neq 0$	$\neq 0$	$\neq 0$	Helices with $\kappa = \frac{-n(c^2+a^2)}{m+ap}$, $\tau = \frac{ma-pc^2}{m+ap}$ and $a \in \mathbb{R} - \{-\frac{m}{p}\}$

m	n	p	Solutions in \mathbb{S}^3 , $C = c^2$
$\neq 0$	$= 0$	$= 0$	Geodesics $\kappa = 0$
$= 0$	$= 0$	$\neq 0$	Circles κ constant and $\tau = 0$
$= 0$	$\neq 0$	$= 0$	Horizontal lifts, via the Hopf map, of curves in \mathbb{S}^2
$\neq 0$	$\neq 0$	$= 0$	Helices with $\kappa = \frac{n(c^2 - \tau^2)}{m}$
$\neq 0$	$= 0$	$\neq 0$	Helices with arbitrary κ and $\tau = \frac{m}{p}$
$= 0$	$\neq 0$	$\neq 0$	Helices with $\kappa = \frac{n(c^2 - a^2)}{ap}$ and $\tau = \frac{c^2}{a}$ and $a \in \mathbb{R} - \{0\}$
$\neq 0$	$\neq 0$	$\neq 0$	Helices with $\kappa = \frac{n(c^2 - a^2)}{m + ap}$, $\tau = \frac{ma + pc^2}{m + ap}$ and $a \in \mathbb{R} - \{-\frac{m}{p}\}$
$\neq 0$	$\neq 0$	$\neq 0$	Lancret curves with $\tau = \frac{p}{n}\kappa - \frac{m}{p}$ and $c = \pm \frac{m}{p}$

Let us make a few remarks on the solutions we have obtained

1. **Euclidean case.** The model $[M(0), \mathcal{F}_{m0p}]$ is also related with the total twist of a Frenet ribbon, [18]. The search of trajectories in the models $[M(0), \mathcal{F}_{mn0}]$ and $[M(0), \mathcal{F}_{m0p}]$, under the additional assumption that they are constrained to lie on a given surface, had been previously considered by L.Santaló, [16]. However, only in the case where such a surface is a round sphere the solution is clear.

From the table corresponding to \mathbb{R}^3 , we see that the model of higher interest is $[M(0), \mathcal{F}_{0np}]$. An algorithm to obtain explicitly, up to motions in \mathbb{R}^3 , all the trajectories of this model works as follows

- (a) We take a plane, say Π in \mathbb{R}^3 and a curve, say $\gamma(u)$, $u \in I \subset \mathbb{R}$, contained in Π .
- (b) Let ξ be a unit vector orthogonal to Π and denote by \mathbf{C}_γ the right cylinder shaped from $\gamma(u)$, that is, the image in \mathbb{R}^3 of the map $\phi : I \times \mathbb{R} \rightarrow \mathbb{R}^3$ defined by

$$\phi(u, v) = \gamma(u) + v \xi.$$

- (c) Let $\gamma_{np}(t)$ be a geodesic of \mathbf{C}_γ with slope θ , $\tan \theta = \frac{n}{p}$, that is

$$\gamma_{np}(t) = \phi(nt, pt) = \gamma(nt) + pt \xi$$

then, $\gamma_{np}(t)$ is a trajectory of the model $[M(0), \mathcal{F}_{0np}]$.

- (d) Moreover, all the trajectories of particles in this model can be obtained in this way. Consequently, up to motions in \mathbb{R}^3 , the set of

trajectories or the moduli space of curves that are solutions to the field equations in the model $[M(0), \mathcal{F}_{0np}]$ can be identified with the space

$$\Gamma_{np} = \{\gamma_{np} : \gamma \text{ is a curve in } \Pi\}.$$

2. **Hyperbolic case.** This is the uninteresting case because the hyperbolic space is free of non-trivial Lancet curves. Therefore, most of the models $[M(-c^2), \mathcal{F}_{mnp}]$ admit a one-parameter family of trajectories which are trivial Lancet curves or helices. The exception to this rule is the model $[M(-c^2), \mathcal{F}_{0n0}]$. That is, this associated with the action measuring the total curvature of trajectories which does not provide any consistent dynamics, (see [2] for more details).
3. **Spherical case.** The most interesting models in the sphere are either $[M(c^2), \mathcal{F}_{0n0}]$ and $[M(c^2), \mathcal{F}_{mnp}]$ with $m.n.p \neq 0$. The former one corresponds again with the action giving the total curvature. In [2], it is showed that the three-dimensional sphere is the only space (no matter the dimension) with constant curvature providing a consistent dynamics for this action. More precisely, the trajectories of this model are nothing but the horizontal lifts, via the usual Hopf map, of arbitrary curves in the two-sphere. It should be noticed that those curves are Lancet ones where the curvature is an arbitrary function while the torsion is nicely determined by the radius of the three-sphere. The later case provides a model which has two kinds of trajectories. First, it has a one-parameter class of trajectories, \mathcal{T} , which are helices and no more comments on it (see table 3). However, the dynamics of this model is completed with a second class of trajectories, \mathcal{T}_{mnp} , that are Lancet curves with

$$\tau = \frac{p}{n} \kappa - \frac{m}{p}.$$

For a better understanding of the family of trajectories \mathcal{T}_{mnp} , we will design an algorithm to obtain its geometric integration.

- (a) First of all, notice that the ratio $\frac{m}{p}$ and the radius, r , of the three-dimensional sphere are constricted to satisfy

$$\frac{p}{m} = \pm r,$$

therefore, without loss of generality we may assume that $r = 1$ and so $m = \pm p$, we will put $m = p$ in the discussion.

- (b) Let consider the usual Hopf map, $\pi : \mathbb{S}^3(1) \rightarrow \mathbb{S}^2(\frac{1}{2})$, between round spheres of radii 1 and $\frac{1}{2}$, respectively. In this setting, π is a Riemannian submersion and the flow of geodesic fibres is generated by a Killing vector field, η , which is sometimes called the *Hopf vector field*.
- (c) If $\beta(u)$, $u \in I \subset \mathbb{R}$, is a curve in $\mathbb{S}^2(\frac{1}{2})$, then $\mathbf{H}_\beta = \pi^{-1}(\beta)$ is a flat surface of $\mathbb{S}^3(1)$ called the *Hopf tube* on β . If $\bar{\beta}$ is a horizontal lift of β , one can use the natural action of the unit circle on $\mathbb{S}^3(1)$ to see that the map, $\psi : I \times \mathbb{R} \rightarrow \mathcal{H}_\beta$, defined by

$$\psi(u, v) = e^{iv} \bar{\beta}(u),$$

is a Riemannian covering map which carries coordinate curves in horizontal lifts of β and fibres, respectively.

- (d) Let $\beta_{np}(t)$ be a geodesic of \mathbf{H}_β with slope θ , $\tan \theta = \frac{n}{p}$, that is

$$\beta_{np}(t) = \psi(nt, pt) = e^{ipt} \bar{\beta}(nt)$$

then, $\beta_{np}(t)$ is a Lancret curve in $\mathbb{S}^3(1)$, which is a trajectory in \mathcal{T}_{mnp} .

- (e) The converse also holds. Every trajectory in \mathcal{T}_{mnp} can be regarded as a geodesic, with slope $\frac{n}{p}$, in a Hopf tube, $\mathbf{H}_\beta = \pi^{-1}(\beta)$, shaped on a curve, β in $\mathbb{S}^2(\frac{1}{2})$. Consequently,

$$\mathcal{T}_{mnp} = \{\beta_{np} : \beta \text{ is a curve in } \mathbb{S}^2(\frac{1}{2})\},$$

recall that $m = \pm p$.

- (f) Since the slope, $\frac{n}{p}$, is known once we choose the action, the space of trajectories, \mathcal{T}_{mnp} , is completely determined, up to congruence, when we give the curvature, in $\mathbb{S}^2(\frac{1}{2})$, of curves β .
- (g) The conclusion is that the space of trajectories and so the dynamics of the particle system $[M(c^2), \mathcal{F}_{mnp}]$ with $m.n.p \neq 0$, is

$$\mathcal{T} \cup \mathcal{T}_{mnp},$$

and so the moduli space of solitons is defined by a couple of parameters, a real number fixing the helix in \mathcal{T} and a smooth function in $C^\infty(I, \mathbb{R})$ which works as the curvature function of a curve in $\mathbb{S}^2(\frac{1}{2})$ which determines the Hopf tube and so the corresponding geodesic with slope $\frac{n}{p}$.

5 Closed trajectories

To study closed trajectories, we will modify a little bit the model $[M(C), \mathcal{F}_{mnp}]$ in the sense that the action, \mathcal{F}_{mnp} , is now assumed to be defined on the space of closed curves in $M(C)$. In this case no boundary conditions are necessary. For obvious reasons, we will restrict ourselves to the spherical case and without loss of generality we will consider the sphere of radius one. Then, we have similar field equations

Proposition 2. *Let \mathcal{C} be the space of immersed closed curves in $\mathbb{S}^3(1)$. The critical points of the variational problem associated with the action $\mathcal{F}_{mnp} : \mathcal{C} \rightarrow \mathbb{R}$ are those closed curves that are solutions of the following Euler-Lagrange equations*

$$(p\tau - m)\kappa + n(1 - \tau^2) = 0, \quad (5)$$

$$p\kappa_s - n\tau_s = 0. \quad (6)$$

It is obvious that the solutions of the above stated field equations are Lancret curves in $\mathbb{S}^3(1)$. Consequently, we need to determine closed Lancret curves in $\mathbb{S}^3(1)$. These trajectories can be characterized according to the following algorithm

1. If we choose a closed curve, $\beta(u)$, $u \in \mathbb{R}$, in $\mathbb{S}^2(\frac{1}{2})$, then, its Hopf tube, $\mathbf{H}_\beta = \pi^{-1}(\beta)$, turn to a flat torus of $\mathbb{S}^3(1)$.
2. The isometry type of a Hopf torus can be determined using the Riemannian covering map, $\psi : \mathbb{R}^2 \rightarrow \mathbf{H}_\beta$, and some well known machinery (see [5], Vol II, p.293, for details and also [12]). In fact, $\mathbf{H}_\beta = \pi^{-1}(\beta)$ is isometric to the \mathbb{R}^2/R , where R is the lattice in \mathbb{R}^2 generated by $(2A, L)$ and $(2\pi, 0)$. Here L denotes the length of β and $A \in (-\pi, \pi)$ the oriented area enclosed by β in the two sphere.
3. Consequently, a Lancret curve of $\mathbb{S}^3(1)$ (recall a geodesic of $\mathbf{H}_\beta = \pi^{-1}(\beta)$) closes if and only if its inverse slope, $\omega = \cot \theta$, satisfies

$$\omega = \frac{1}{L}(2A + q\pi),$$

where q is a rational number.

4. On the other hand, $\gamma \in \mathcal{C}$ is a trajectory of $[M(C), \mathcal{F}_{mnp}]$ if and only if its inverse slope satisfies $\omega = \frac{p}{n}$. In particular, it closes if and only if its inverse slope, $\omega = \frac{p}{n}$, satisfies the following quantization principle

$$\frac{p}{n}L - 2A \quad \text{is a rational multiple of } \pi.$$

6 Existence of closed trajectories

For simplicity, we can assume the area A to be positive, changing if necessary the orientation of β . The only further restriction on (A, L) to define an embedded closed curve in the two sphere is given by the iso-perimetric inequality in $\mathbb{S}^2(\frac{1}{2})$, which can be written as

$$L^2 + 4A^2 - 4\pi A \geq 0.$$

In terms of $(2A, L)$, this inequality is expressed as

$$L^2 + (2A - \pi)^2 \geq \pi^2.$$

In the $(L, 2A)$ -plane, we define the region

$$\Delta = \{(L, 2A) : L^2 + (2A - \pi)^2 \geq \pi^2 \text{ and } 0 \leq A \leq \pi\},$$

then for each point $z \in (L, 2A) \in \Delta$ there is an embedded closed curve, β^z , in $\mathbb{S}^2(\frac{1}{2})$ with length L and enclosed area A . We already know that a geodesic, β_{np}^z , of $\mathbf{H}_{\beta^z} = \pi^{-1}(\beta^z)$ with slope $\frac{1}{\omega} = \frac{n}{p}$ is a trajectory of the model $[M(C), \mathcal{F}_{mnp}]$. Moreover, we use the quantization principle to see that it closes if and only if the straight line, in the $(L, 2A)$ -plane, with slope $\omega = \frac{p}{n}$ cuts the $2A$ -axis at a height which is a rational multiple of π .

Theorem. *For any couple of parameters, n and p with $n.p \neq 0$, there exists an infinite series of closed trajectories in the model $[M(C), \mathcal{F}_{mnp}]$ on the three-dimensional sphere of radius $r = \frac{p}{m}$, $m \neq 0$. This series includes all the geodesics β_{np}^z in $M_{\beta^z} = \pi^{-1}(\beta^z)$ with slope $\omega = \frac{p}{n}$ and β^z determined as above by $z = (L, 2A)$ in the following region*

$$\Delta \cap \left(\bigcup_{q \in \mathbb{Q}} \left(\frac{p}{n}L - 2A = q\pi \right) \right).$$

Remark. A quantization principle to characterize the moduli sub-space of closed trajectories in \mathcal{T} can be also obtained. In this case, since the trajectories are helices, then they are geodesics of Hopf tubes (Hopf tori to be closed) shaped on geodesic circles in $\mathbb{S}^2(\frac{1}{2})$. Moreover the slope in the corresponding flat torus depends on the parameter a (see Table 3) according to

$$\omega = \frac{p - m}{n(1 + a)}.$$

7 Conclusions

The events of this note take place in $D = 3$ spaces with constant curvature, $M(C)$. In this setting, we have considered models for relativistic particles where the Lagrangian densities depend linearly on both the curvature and the torsion of the trajectories.

The moduli spaces of classical solutions are completely and explicitly obtained. A part of these spaces in flat backgrounds was known, solutions being helices. However, the more interesting models are those where non helicoidal solutions appear. In these cases the solutions are non trivial Lancret curves in flat spaces and spherical ones, respectively.

The complete spaces of solutions are formally described in three tables. However, we design algorithms providing the geometric integrations of these spaces of solutions. The geometry of Lancret curves in the classical setting so as its extension to spherical framework, based in the the Hopf map, are the chief points in these algorithms.

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